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THE INCOMPLETE AND COMPLETE r-CENTRAL LAH-BELL POLYNOMIALS

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ABSTRACT. Both the complete and incomplete Bell polynomials are multivariate forms for Bell polynomials and Stirling numbers of the second kind. These polynomials are a crucial role in enumerative combinatorics. Recently, H.K. Kim introduced both the central Lah numbers and r-central Lah numbers. With this in mind, this paper will be subdivided into two parts to address the above mathematics concepts: in the first part, that will be a question of introducing both complete and incomplete central Lah-Bell polynomials, as multivariate forms of the central Lah numbers and the central Lah-Bell polynomials, respectively. We give a relation between the central Lah-Bell numbers and the complete Bell polynomials by using the Kölbig-Coeffey equation. This entail deriving explicit formulas for these polynomials and numbers, as well as conducting research into some identities for these polynomials. In the second part, both the r-central complete and incomplete Lah-Bell polynomials as multivariate forms of both the r-central Lah-numbers and the r-extended Lah-Bell polynomials are introduced and also are derive explicit formulas and several noble identities.

1. INTRODUCTION

The unsigned Lah-number counts the number of partitions of a set with $1, 2, \cdots, n$ elements into k ordered blocks with no box left empty. The Lah numbers appears non-crossing partitions, Dyck paths, q-analogues as well as falling and rising factorials [6, 9, 15, 16, 23, 24]. The r-Lah number ($r \in \mathbb{N}$) counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks [16, 27, 28]. The central factorial numbers, introduced by Riordan [30], are often appears in their properties and applications to difference calculus, spline theory, and to approximation theory, etc. [3-6, 10-14, 17, 19, 20, 30, 32]. Recently, H.K. Kim introduced and studied the central Lah numbers and central Lah-Bell numbers [13], and the r-central Lah numbers and r-central Lah-Bell numbers ($r \in \mathbb{N}$) [14]. As well known, both the complete and incomplete Bell polynomials are multivariate forms for Bell polynomials and Stirling numbers of the second kind, respectively. These polynomials are often used in combinatorics, statistic and mathematical applications [1, 3-8, 25, 26, 29]. With these points in mind, we study two types of the complete and incomplete Bell polynomials related problems in this paper. In Section 2, both the complete and incomplete central Lah-Bell polynomials, as multivariate forms of the central lah-numbers and the central Lah-Bell polynomials are introduced. We research explicit formulas for these polynomials and numbers, as well as combinatorial identities for these polynomials. In Section 3, both the complete and incomplete r-central Lah-Bell polynomials as multivariate forms of both the r-central Lah-numbers and the r-central Lah-Bell polynomials are introduced. In respect to both the numbers and polynomials, we also derive explicit formulas and several properties.

First, we give some definitions and properties needed in this paper.

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For $n \geq k \geq 0$, the Stirling numbers of the second kind $S_2(n,k)$ is the number of ways to partition a set with n elements into k non-empty subsets, and the n-th Bell number

$$B_n = \sum_{k=0}^{n} S_2(n,k), \quad \text{(see [1, 2, 6])}.$$

As known well, the generating function of Bell polynomials is given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \ (n \geq 0), \quad \text{(see [1, 6, 18, 25])}, \tag{1}$$

where $B_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k$ are called the bell polynomials.

As the multivariate version of the Bell polynomial $B_n(x)$, the generating function of the complete exponential Bell polynomials $B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$ are given by

$$\exp\big(\sum_{h=1}^{\infty} \alpha_h \frac{t^h}{h!}\big) = \sum_{n=0}^{\infty} B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)\frac{t^n}{n!}, \quad \text{(see } [1, 6-8, 21]), \tag{2}$$

where $\exp(t) = e^t$.

From (2), it is easy to see that the complete exponential Bell polynomials are

$$B_n(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{h_1+2h_2+\cdots+nh_n=n} \frac{n!}{h_1!h_2!\cdots h_n!}\left(\frac{\alpha_1}{1}\right)^{h_1}\left(\frac{\alpha_2}{2!}\right)^{h_2}\cdots\left(\frac{\alpha_n}{n!}\right)^{h_n}, \quad \text{(see } [6-8, 21]), \tag{3}$$

where the sum over all nonnegative integers $h_1, h_2, \cdots, h_n$ satisfying $h_1 + 2h_2 + \cdots + nh_n = n$.

Combining by (1) and (2), we observe that

$$\sum_{n=0}^{\infty} B_n(x, x, \cdots, x)\frac{t^n}{n!} = \exp\left(\sum_{h=1}^{\infty} x\frac{t^h}{h!}\right) = \exp(x(e^t-1)) = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{4}$$

From (4), we have $B_n(x, \cdots, x) = B_n(x)$, where $B_n(x)$ are the ordinary Bell polynomials.

As the multivariate version of the Stirling numbers $S(n,k)$ of the second kind, the generation function of the incomplete Bell polynomials $B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})$ is given by

$$\frac{1}{k!}\left(\sum_{h=1}^{\infty} x_h\frac{t^h}{h!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(\alpha_1, \alpha_2, \cdots, \alpha_{n-k+1})\frac{t^n}{n!}, \quad \text{(see [1, 7, 8, 21])}. \tag{5}$$

From (5), we obtain

$$\begin{aligned} &B_{n,k}(\alpha_1, \alpha_2, \cdots, \alpha_{n-k+1}) \\ &= \sum_{h_1+2h_2+\cdots+nh_n=n} \frac{n!}{h_1!h_2!\cdots h_{n-k+1}!}\left(\frac{\alpha_1}{1}\right)^{h_1}\left(\frac{\alpha_2}{2!}\right)^{h_2}\cdots\left(\frac{\alpha_{n-k+1}}{(n-k+1)!}\right)^{h_{n-k+1}}, \quad \text{(see [7, 8, 21])}, \end{aligned} \tag{6}$$

where the sum over all nonnegative integers $h_1, h_2, \cdots, h_n$ satisfying $h_1 + h_2 + \cdots + h_{n-k+1} = k$ and $h_1 + 2h_2 + \cdots + (n-k+1)h_{n-k+1} = n$.

For a C^∞- function, the higher-order derivatives of $e^{f(x)}$ can be expressed in terms of the complete Bell polynomials $B_n(x_1, x_2, \cdots, x_n)$ which is known as Kölbig-coeffey equation given as in the following:

$$\frac{d^n}{dx^n}e^{f(x)} = e^{f(x)}B_n(f^{(1)}(x), f^{(2)}(x), \cdots, f^{(n)}(x)) \quad (m \geq 1), \quad \text{(see [21, 22])}. \tag{7}$$

It is well-known that

$$(8)\qquad (1-t)^{-m}=\sum_{l=0}^{\infty}\binom{-m}{l}(-1)^l t^l=\sum_{l=0}^{\infty}<m>_l\frac{t^l}{l!},\quad \text{(see [6])}.$$

where $<x>_0=1$ and $<x>_n=x(x+1)(x+2)\cdots(x+n-1)$, $(n\geq 1)$.

The central factorial $x^{[n]}$ is defined by

$$(9)\quad x^{[0]}=1,\ x^{[n]}=x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2))\cdots(x-\frac{n}{2}+1),\quad (n\geq 1),\quad \text{(see [3, 4, 14, 28])}.$$

Riordan [30] showed that the central factorial numbers of the second kind $T(n,k)$ are the coefficients in the expansion of x^n in terms of central factorials given by

$$x^n=\sum_{k=0}^{n}T(n,k)x^{[k]},\quad (n\geq k\geq 0)\quad \text{(see [17, 19, 30])}.$$

It is easy to see that the generating function of $T(n,k)$ is

$$(10)\qquad \frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!},\quad \text{(see [17, 19, 30])}.$$

From (10), the generating function for the central Bell polynomials $B_n^{(c)}(x)$ is given by

$$(11)\qquad e^{x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}=\sum_{n=0}^{\infty}B_n^{(c)}(x)\frac{t^n}{n!},\quad \text{(see [17, 19])},$$

where $B_n^{(c)}(x)=\sum_{k=0}^{n}T(n,k)x^k$ are the central polynomials Bell polynomials and $B_n^{(c)}=B_n^{(c)}(1)$ are the central Bell numbers.

As the multivariate versions of the central factorial numbers of the second kind $T(n,k)$ and the central Bell polynomials $B_n^{(c)}(x)$, Kim et al. introduced the generating functions of the central incomplete Bell polynomials $T_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})$ and the central complete Bell polynomials $B_n(c)(x|\alpha_1,\alpha_2,\cdots,\alpha_n)$ respectively as follows:

$$(12)\quad \frac{1}{k!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)\frac{t^h}{h!}\right)^k=\sum_{n=k}^{\infty}T_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})\frac{t^n}{n!},\quad \text{(see [17, 18])}.$$

and

$$(13)\quad \exp\left(x\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)\frac{t^h}{h!}\right)=\sum_{n=0}^{\infty}B_n^{(c)}(x|\alpha_1,\alpha_2,\cdots,\alpha_n)\frac{t^n}{n!}\quad \text{(see [17, 18])}.$$

For $n,k\geq 0$ with $n-k\equiv 0(mod2)$, combining by (6) and (12), we get

(14)

$$T_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})$$

$$=\sum\frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\left(\frac{\alpha_1}{1!}\right)^{i_1}\left(\frac{0}{2\cdot 2!}\right)^{i_2}\left(\frac{\alpha_3}{2^2\cdot 3!}\right)^{i_3}\cdots\left(\frac{\alpha_{n-k+1}}{2^{n-k}(n-k+1)!}\right)^{i_{n-k+1}},\quad \text{(see [17-19])},$$

where the summation is over all integers $i_1,i_2,\cdots,i_{n-k+1}\geq 0$ such that $i_1+\cdots+i_{n-k+1}=k$ and $i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n$.

For $n\geq k\geq 0$ with $n-k\equiv 0(mod2)$, by (6) and (14), we note that

$$(15)\qquad T_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})=B_{n,k}\left(\alpha_1,0,\frac{\alpha_3}{2^2},0,\cdots,\frac{\alpha_{n-k+1}}{2^{n-k}}\right),\quad \text{(see [17-19])}.$$

2. The central incomplete and complete Lah-Bell polynomials

In this section, we consider the incomplete central Lah-Bell polynomials and the complete central Lah-Bell polynomials respectively as the multivariate version of the central Lah-numbers and the central Lah-Bell polynomials, respectively. We derive explicit formulas for these polynomials and numbers, and study relations between these polynomials and the central Lah-numbers and the central Lah-Bell polynomials.

The unsigned Lah-number $L(n,k)$ counts the number of partitions of a set with $1,2,\cdots,n$ elements into k ordered blocks with no box left empty. Lah-numbers are rarely called Stirling numbers of the third kind. It is well known that an explicit formula of the Lah numbers $L(n,k)$ and the generating function of these respectively are given by

$$L(n,k)=\frac{n!}{k!}\binom{n-1}{k-1}, \quad \text{(see [6, 9, 13, 23, 24])}. \tag{16}$$

and

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L(n,k)\frac{t^n}{n!}, \quad \text{(see [6, 9, 13, 23, 24])}. \tag{17}$$

The Lah-Bell numbers LB_n and the Lah-Bell polynomials $LB_n(x)$ respectively are given by

$$LB_n=\sum_{k=0}^{n}L(n,k), \text{ and } \quad LB_n(x)=\sum_{k=0}^{n}L(n,k)x^k, \quad (n\geq 0), \quad \text{(see [9])}. \tag{18}$$

As known well, the generating function of Lah-Bell polynomials $LB_n(x)$ is

$$e^{x(\frac{t}{1-t})}=\sum_{n=0}^{\infty}LB_n(x)\frac{t^n}{n!}, \quad (n\geq 0), \quad \text{(see [9])}. \tag{19}$$

When $x=1$, $LB_n=LB_n(1)$ are Lah-Bell numbers.

In view of (10) and (11), H.K. Kim introduced the central Lah numbers $L^{(C)}(n,k)$ and the central Lah-Bell numbers $LB_n^{(C)}$ given by the generating functions respectively

$$\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k=\sum_{n=k}^{\infty}L^{(C)}(n,k)\frac{t^n}{n!}, \quad \text{(see [13])}, \tag{20}$$

and

$$exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)=\sum_{n=0}^{\infty}LB_n^{(C)}\frac{t^n}{n!}, \quad \text{(see [13])}, \tag{21}$$

where $LB_n^{(C)}=\sum_{k=0}^{n}L^{(C)}(n,k)$. Furthermore, H.K. Kim showed that the central Lah numbers represent as Riemann integrals in [13].

For $n\geq k\geq 0$, an explicit formula of Lah numbers $L^{(C)}(n,k)$ is

$$L^{(C)}(n,k)=\sum_{i=0}^{k}\sum_{l=k-i}^{n}(-1)^{k-i+l}\left(\frac{1}{2}\right)^nL(n-l,i)L(l,k-i), \quad \text{(see [13])},$$

where $L(0,0)=1,\ L(n,0)=0$, and $L(n,k)=0$ for all $k>n$.

In view of (19), H.K. Kim considered the central Lah-Bell polynomials $LB_n^{(C)}(x)$ by

$$LB_n^{(C)}(x)=\sum_{k=0}^{n}L^{(C)}(n,k)x^k, \quad (n\geq 0), \tag{22}$$

when $x=1$, $LB^{(C)}(1) := LB^{(C)}$ are called the central Lah-Bell numbers.

For $n \geq k \geq 0$, the generating function of the central Lah-Bell polynomials $LB_n^{(C)}(x)$ is

$$\exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right) = \sum_{n=0}^{\infty} LB_n^{(C)}(x)\frac{t^n}{n!}, \quad \text{(see [13]).} \tag{23}$$

The next theorem give a relation between the central Lah-Bell numbers and the complete Bell polynomials by using the Kölbig-Coeffey equation (8).

Theorem 1. *For $n \geq 1$, when n is odd integer, we have*

$$\begin{aligned} LB_n^{(C)} &= B_n\left(1,0,\frac{3!}{2^2},0,\cdots,0,\frac{n!}{2^{n-1}}\right) \\ &= \sum_{h_1+3h_3+\cdots+nh_n=n} \frac{n!}{h_1!h_3!\cdots h_{n-2}!h_n!}\left(\frac{1}{2}\right)^{2h_3+4h_5+\cdots+(n-1)h_n}. \end{aligned}$$

When n is even integer, we have

$$\begin{aligned} LB_n^{(C)} &= B_n\left(1,0,\frac{3!}{2^2},0,\cdots,\frac{(n-1)!}{2^{n-2}},0\right) \\ &= \sum_{h_1+3h_3+\cdots+(n-1)h_{n-1}=n} \frac{n!}{h_1!h_3!\cdots h_{n-1}!}\left(\frac{1}{2}\right)^{2h_3+4h_5+\cdots+(n-2)h_{n-1}}. \end{aligned}$$

Proof. Let $f(t)=2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)$. Then we observe that

$$f^{(n)}(t) = \frac{d^n}{dt^n}f(t) = 2\left(\frac{n!}{(2-t)^{n+1}}-\frac{(-1)^n\, n!}{(2+t)^{n+1}}\right). \tag{24}$$

Combining (4) and the Kölbig-Coeffey equation (8), when n is odd integer, we obtain

$$\frac{d^n}{dt^n}\exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\bigg|_{t=0} = B_n\left(1,0,\frac{3!}{2^2},0,\cdots,0,\frac{n!}{2^{n-1}}\right), \tag{25}$$

and when n is even integer,

$$\frac{d^n}{dt^n}\exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\bigg|_{t=0} = B_n\left(1,0,\frac{3!}{2^2},0,\cdots,\frac{(n-1)!}{2^{n-2}},0\right). \tag{26}$$

On the other hand, by (23), we obtain

$$\frac{d^n}{dt^n}\exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\bigg|_{t=0} = \frac{d^n}{dt^n}\sum_{l=0}^{\infty} LB_l^{(C)}\frac{t^l}{l!}\bigg|_{t=0} = LB_n^{(C)}. \tag{27}$$

Therefore, from (25), (26) and (27), we arrive at the desired results.

□

Now, we define the incomplete central Lah-Bell polynomials $H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})$ which are given by

$$\frac{1}{k!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)^k = \sum_{n=k}^{\infty} H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})\frac{t^n}{n!}. \tag{28}$$

From (6), (14) and (28), the following theorem can be easily obtained an an explicit formula of the incomplete central Lah-Bell polynomials.

Theorem 2. *For $n,k \geq 0$ with $n-k \equiv 0(mod 2)$, we have*

$$\begin{aligned} H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1}) &= T_{n,k}(1!\alpha_1,2!\alpha_2,\cdots,(n-k+1)!\alpha_{n-k+1}) \\ &= B_{n,k}\left(\alpha_1,0,\frac{3!}{2^2}\alpha_3,0,\cdots,\frac{(n-k+1)!}{2^{n-k}}\alpha_{n-k+1}\right) \\ &= \sum \frac{n!}{h_1!,h_3!\cdots,h_{n-k+1}!}(\alpha_1)^{h_1}\left(\frac{\alpha_3}{2^2}\right)^{h_3}\cdots\left(\frac{\alpha_{n-k+1}}{2^{n-k}}\right)^{h_{n-k+1}}, \end{aligned}$$

where the summation is over all integers $h_1,h_2,\cdots,h_{n-k+1} \geq 0$ such that $h_1+h_2+\cdots+h_{n-k+1}=k$ and $h_1+2h_2+\cdots+(n-k+1)h_{n-k+1}=n$.

Remark. In Theorem 2, we observe that

$$\begin{aligned} H_{n,k}\left(\alpha_1,\frac{2}{2!}\alpha_2,\frac{2^2}{3!}\alpha_3,\cdots,\frac{2^{n-k}}{(n-k+1)!}\alpha_{n-k+1}\right) &= T_{n,k}(\alpha_1,2\alpha_2,\cdots,2^{n-k+1}\alpha_{n-k+1}) \\ &= B_{n,k}(\alpha_1,0,\alpha_3,0,\cdots,\alpha_{n+k-1}). \end{aligned}$$

For example, from the example in section 3 of [8], we get

$$\begin{aligned} &H_{11,7}\left(\frac{1}{1!}\alpha_1,\frac{2}{2!}\alpha_2,\frac{2^2}{3!}\alpha_3,\frac{2^3}{4!}\alpha_4,\frac{2^4}{5!}\alpha_5,\frac{2^5}{6!}\alpha_6,\frac{2^6}{7!}\alpha_7\right) \\ &\qquad = B_{11,7}(\alpha_1,0,\alpha_3,0,\cdots,\alpha_{n+k-1}) = 4620\alpha_1^5\alpha_3^2+426\alpha_1^6\alpha_5. \end{aligned}$$

That is, there are two ways of partitioning a set with 11 elements into 7 ordered blocks with odd sizes, and 4620 partitions with ordered blocks of size $,1,1,1,1,1,3,3$ and 426 partitions with ordered blocks of size $1,1,1,1,1,1,5$.

The next theorem give a relation between the central Lah numbers and the incomplete central Lah-Bell polynomials

Theorem 3. *For $n,k \geq 0$ with $n-k \equiv 0(mod 2)$, we have*

$$H_{n,k}(x,x^2,x^3,\cdots,x^{n-k+1}) = x^n L^{(C)}(n,k). \tag{29}$$

In particular,

$$H_{n,k}(1,1,\cdots,1) = L^{(C)}(n,k), \tag{30}$$

where $L^{(C)}(n,k)$ are the central Lah numbers.

Proof. For $n \geq k \geq 0$ with $n-k \equiv 0(mod 2)$, by (12) and (28), we get

$$\begin{aligned} \sum_{n=k}^{\infty} H_{n,k}(x,x^2,x^3,\cdots,x^{n-k+1})\frac{t^n}{n!} &= \frac{1}{k!}(xt+\frac{x^3}{2^2}t^3+\frac{x^5}{2^4}t^5+\cdots)^k \\ &= \frac{1}{k!}\left(\frac{\frac{t}{2}xt}{1-\frac{t}{2}xt}-\frac{(-\frac{t}{2}xt)}{1-(-\frac{t}{2}xt)}\right)^k \\ &= \sum_{n=k}^{\infty} L^{(C)}(n,k)\frac{(tx)^n}{n!} = \sum_{n=k}^{\infty} x^n L^{(C)}(n,k)\frac{t^n}{n!}. \end{aligned} \tag{31}$$

Therefore, by comparing with coefficients of both side of (31), we obtain the identity (29). When $x=1$, we have the identity (30) immediately. □

By the definition of the incomplete central Lah-Bell polynomials, for $n \geq k \geq 0$ with $n-k \equiv 0(mod 2)$, we obtain

$$H_{n,k}(x,x,\cdots,x) = x^k H_{n,k}(1,1,\cdots,1) = x^k L^{(C)}(n,k), \tag{32}$$

and

$$H_{n,k}(z\alpha_1, z\alpha_2, \cdots, z\alpha_{n-k+1}) = z^k H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1}). \tag{33}$$

Now, we naturally define the complete central Lah-Bell polynomials $H_n^{(C)}(x|\alpha_1,\alpha_2,\cdots,\alpha_n)$ by

$$\exp\left(x\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right) = \sum_{n=0}^{\infty} H_n^{(C)}(x|\alpha_1,\alpha_2,\cdots,\alpha_n)\frac{t^n}{n!}. \tag{34}$$

When $x=1$,

$$H_n^{(C)}(1|\alpha_1,\alpha_2,\cdots,\alpha_n) = H_n^{(C)}(\alpha_1,\alpha_2,\cdots,\alpha_n) \tag{35}$$

are called the complete central Lah-Bell numbers.

For $n \geq k \geq 0$ with $n-k \equiv 0(mod 2)$, from (28), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(C)}(x|\alpha_1,\alpha_2,\cdots,\alpha_n)\frac{t^n}{n!} &= \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\sum_{n=k}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)^k \\ &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})\right)\frac{t^n}{n!}.\end{aligned} \tag{36}$$

Thus, from (36), we get

$$H_n^{(C)}(x|\alpha_1,\alpha_2,\cdots,\alpha_n) = \sum_{k=0}^{n} x^k H_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1}). \tag{37}$$

The next theorem is a relation between the central Lah-Bell polynomials and the incomplete central Lah-Bell polynomials by using the complete central Lah-Bell polynomials.

Theorem 4. *For $n \geq k \geq 0$, we have*

$$LB_n^{(C)}(x) = \begin{cases} H_n^{(C)}(x|1,1,\cdots,1) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

where $LB_n^{(C)}(x)$ are the central Lah-Bell polynomials.

Proof. From (23) and (34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(C)}(x|1,1,\cdots,1)\frac{t^n}{n!} &= \exp\left(x\sum_{h=1}^{\infty}\frac{1}{2^h}(1-(-1)^h)t^h\right) \\ &= \exp\left(x(t+\frac{1}{2^2}t^3+\frac{1}{2^4}t^5+\cdots)\right) \\ &= \exp\left(x\left(\frac{\frac{t}{2}}{1-\frac{t}{2}} - \frac{-\frac{t}{2}}{1-(-\frac{t}{2})}\right)\right) = \sum_{n=0}^{\infty} LB_n^{(C)}(x)\frac{t^n}{n!}.\end{aligned} \tag{38}$$

By comparing with coefficients of both sides of (38), we arrive at the desired result. □

Theorem 5. *For $n,k \geq 0$ with $n-k \equiv 0(mod2)$, we have*

$$H_n^{(C)}(x|1,1,\cdots,1) = \sum_{k=0}^{n} x^k B_{n,k}\left(1!,0,\frac{3!}{2^2},0\cdots,\frac{(n-k+1)!}{2^{n-k}}\right),$$

where $B_{n,k}(\alpha_1,\alpha_2,\cdots,\alpha_{n-k+1})$ are the incomplete Bell polynomials.

Proof. For $n \geq k \geq 0$ with $n-k \equiv 0(mod2)$, from (37) and Theorem 2, we get

$$\begin{aligned} H_n^{(C)}(x|1,1,\cdots,1) &= \sum_{k=0}^{n} x^k H_{n,k}(1,1,\cdots,1) \\ &= \sum_{k=0}^{n} x^k T_{n,k}(1!,2!,\cdots,(n-k+1)!) \\ &= \sum_{k=0}^{n} x^k B_{n,k}\left(1!,0,\frac{3!}{2^2},0\cdots,\frac{(n-k+1)!}{2^{n-k}}\right). \end{aligned} \tag{39}$$

By (39), we have the desired result. □

The next theorem give an explicit formula of the complete central Lah-Bell polynomials.

Theorem 6. *For $n \geq k \geq 0$ with $n-k \equiv 0(mod2)$, the explicit formula of the complete central Lah-Bell numbers is*

$$\begin{aligned} &H_n^{(C)}(\alpha_1,\alpha_2,\cdots,\alpha_n) \\ &\qquad = n! \sum_{h_1+3h_3+\cdots+nh_n=n} \frac{1}{h_1!h_2!\cdots h_n!}(\alpha_1)^{h_1}\left(\frac{\alpha_3}{2^2}\right)^{h_3}\cdots\left(\frac{\alpha_n}{2^{n-1}}\right)^{h_n}. \end{aligned}$$

Proof. From (34), we observe that

(40)
$$
\begin{aligned}
\sum_{n=0}^{\infty} H_n^{(C)}(1|\alpha_1,\alpha_2,\cdots,\alpha_n)\frac{t^n}{n!} &= \exp\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)\\
&= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)^n\\
&= 1+\frac{1}{1!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)+\frac{1}{2!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)^2\\
&\quad +\frac{1}{3!}\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)^3+\cdots\\
&= 1+\frac{1}{1!}\left(\alpha_1 t+\frac{1}{2^2}\alpha_3t^3+\frac{1}{2^4}\alpha_5t^5+\cdots\right)+\frac{1}{2!}\left(\alpha_1 t+\frac{1}{2^2}\alpha_3t^3+\frac{1}{2^4}\alpha_5t^5+\cdots\right)^2\\
&\quad +\frac{1}{3!}\left(\alpha_1 t+\frac{1}{2^2}\alpha_3t^3+\frac{1}{2^4}\alpha_5t^5+\cdots\right)^3+\cdots\\
&= 1+\frac{1}{1!}\alpha_1+\frac{1}{2!}\alpha_1^2t^2+\left(\frac{1}{3!}\alpha_1^3+\frac{1}{2^2}\alpha_3\right)t^3+\frac{1}{4!}\alpha_1^4t^4+\cdots\\
&= \sum_{n=0}^{\infty}\Bigg(\sum_{h_1+2h_2+\cdots+nh_n=n}\frac{1}{h_1!h_2!\cdots h_n!}\\
&\qquad\times\left(\frac{\alpha_1}{2^0}\right)^{h_1}\left(\frac{0}{2^1}\right)^{h_2}\left(\frac{\alpha_3}{2^2}\right)^{h_3}\cdots\left(\frac{\alpha_n(1-(-1)^n)}{2^{n-1}}\right)^{h_n}\Bigg)t^n.
\end{aligned}
$$

By comparing with coefficients of both side of (40), we have the desired result. □

3. The complete and incomplete r-central Lah-Bell polynomials

In this section, we introduce r-extended central incomplete Lah-Bell polynomials and r-extended central complete Lah-Bell polynomials respectively as the multivariate version of the r-extended central Lah-Bell polynomials and r-extended central Lah-numbers ($r \in \mathbb{N}$). We derive explicit formulas for these polynomials and numbers, and study relations between these polynomials and the complete and incomplete Bell polynomials.

First, we give some definitions and properties needed in this section.

Throughout in this section, let $r \in \mathbb{N}$.

Mihoubi and Rahman [26] introduced the incomplete r-Bell polynomials given by the generating function

(41)
$$\frac{1}{k!}\left(\sum_{h=1}^{\infty}\alpha_h\frac{t^h}{h!}\right)^k\left(\sum_{m=0}^{\infty}\upsilon_{m+1}\frac{t^m}{m!}\right)^r=\sum_{n\geq k}B_{n+r,k+r}^{(r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!},\quad \text{(see [11, 12, 26])}.$$

Thus, from (41), we observe that

(42)
$$
\begin{aligned}
&B_{n+r,k+r}^{(r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\\
&=\sum\left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{\alpha_1}{1!}\right)^{k_1}\left(\frac{\alpha_2}{2!}\right)^{k_2}\cdots\right]\left[\frac{r!}{r_0!r_1!r_2!\cdots}\left(\frac{\upsilon_1}{0!}\right)^{r_0}\left(\frac{\upsilon_2}{1!}\right)^{r_1}\left(\frac{\upsilon_3}{2!}\right)^{r_3}\cdots\right],
\end{aligned}
$$

where the summation is over all integers $k_1,k_2,\cdots \geq 0$ and $r_0,r_1,r_2\cdots \geq 0$, such that

$$\sum_{i\geq 1} k_i = k, \quad \sum_{h\geq 0} r_i = r \quad \text{and} \quad \sum_{i\geq 1} i(k_i + r_i) = n.$$

When $\alpha_1, \alpha_2, \cdots$, and $\upsilon_1, \upsilon_2, \cdots$ are any sequences of nonnegative integers, Mihoubi and Rahman gave also some combinatorial interpretations in terms of set partitions in [26].

Naturally, the generating function of the complete r-Bell polynomials are given by

(43)
$$\exp\left(\sum_{h=1}^{\infty} \alpha_h \frac{t^h}{h!}\right)\left(\sum_{m=0}^{\infty} \upsilon_{m+1}\frac{t^m}{m!}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\alpha_1, \alpha_2, \cdots : \upsilon_1, \upsilon_2, \cdots)\frac{t^n}{n!}, \quad (r \in \mathbb{R}), \quad (\text{see } [11, 12, 26]).$$

Kim et al. introduced the r-extended central factorial numbers of the second kind given by

$$\frac{1}{k!} e^{rt}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T_r(n+r, k+r)\frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see } [11, 12]). \tag{44}$$

In view of (11), Kim et al. also introduced the r-extended central Bell polynomials $B_n^{(c,r)}(x)$ associated the r-extended central factorial numbers of the second kind are given by

$$e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)} = \sum_{n=0}^{\infty} B_n^{(c,r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}), \quad (\text{see } [11, 12]). \tag{45}$$

For $n \geq 0$, we note that

$$B_n^{(c,r)}(x) = \sum_{k=0}^{n} x^k T_r(n+r, k+r), \quad (r \in \mathbb{R}), \quad (\text{see } [11, 12]). \tag{46}$$

Kim et al. defined the r-extended central incomplete Bell polynomials by

$$\begin{aligned} &\frac{1}{k!}\left(\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h (\alpha_h - (-1)^h \alpha_h)\frac{t^h}{h!}\right)^k \left(\sum_{m=0}^{\infty} \upsilon_{i+1}\frac{t^m}{m!}\right)^r \\ &\qquad = \sum_{n=k}^{\infty} T_{n+r,k+r}^{(r)}(\alpha_1, \alpha_2, \cdots : \upsilon_1, \upsilon_2, \cdots)\frac{t^n}{n!}, \quad (\text{see } [11, 12]), \end{aligned} \tag{47}$$

for any $k \in \mathbb{N}\bigcup\{0\}$.

For $n, k \geq 0$ with $n \geq k$, by (47), it is easy to see that

$$\begin{aligned} T_{n+r,k+r}^{(r)}(\alpha_1, \alpha_2, \cdots : \upsilon_1, \upsilon_2, \cdots) = \sum &\left[\frac{n!}{k_1!k_3!k_3!\cdots}\left(\frac{\alpha_1}{1!}\right)^{k_1}\left(\frac{\alpha_3}{2^2 3!}\right)^{k_3}\left(\frac{\alpha_5}{2^4 5!}\right)^{k_5}\cdots\right] \\ &\times \left[\frac{r!}{r_0!r_1!r_2!\cdots}\left(\frac{\upsilon_1}{0!}\right)^{r_0}\left(\frac{\upsilon_2}{1!}\right)^{r_1}\left(\frac{\upsilon_3}{2!}\right)^{r_2}\cdots\right], \end{aligned} \tag{48}$$

where the summation is over all integers $k_1, k_3, k_5, \cdots \geq 0$ and $r_0, r_1, r_2 \cdots \geq 0$, such that

$$\sum_{i\geq 1} k_{2i-1} = k, \quad \sum_{i\geq 0} r_i = r \quad \text{and} \quad \sum_{i\geq 1}(2i-1)k_{2i-1} + \sum_{i\geq 1} i r_i = n.$$

Combining with (42) and (48), we observe that

$$T_{n+r,k+r}^{(r)}(\alpha_1, \alpha_2, \cdots : \upsilon_1, \upsilon_2, \cdots) = B_{n+r,k+r}^{(r)}(\alpha_1, 0, \frac{\alpha_3}{2^2}, 0, \cdots : \upsilon_1, \upsilon_2, \upsilon_3, \cdots). \tag{49}$$

The r-Lah number $L_r(n,k)$ counts the number of partitions of a set with $n+r$ elements into $k+r$ ordered blocks such that r distinguished elements have to be in distinct ordered blocks and an explicit formula of $L_r(n,k)$ given by

$$L_r(n,k) = \binom{n+2r-1}{k+2r-1}\frac{n!}{k!}\ (k \geq 0), \qquad \text{(see [14, 16, 27, 28])}. \tag{50}$$

From (50), we have the generating function of $L_r(n,k)$ given by

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k \left(\frac{1}{1-t}\right)^{2r} = \sum_{n=k}^{\infty} L_r(n,k)\frac{t^n}{n!}, (k \geq 0), \qquad \text{(see [14, 16, 27, 28])}. \tag{51}$$

Recently, in view of (20) and (44), H.K. Kim introduced the generating function of the central r-Lah numbers $L_r^{(C)}(n,k)$ given by

$$\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k \left(\frac{1}{1-t}\right)^{2r} = \sum_{n=k}^{\infty} L_r^{(C)}(n,k)\frac{t^n}{n!}, \qquad \text{(see [14])}. \tag{52}$$

For $n \geq k \geq 0$, an explicit formula of the central r-Lah-numbers $L_r^{(C)}(n,k)$ is

$$L_r^{(C)}(n,k) = \sum_{m=0}^{n}\sum_{i=0}^{k}\sum_{l=k-i}^{m}\binom{n}{m}(-1)^{k-i+l}\left(\frac{1}{2}\right)^m < 2r >_{n-m} L(m-l,i)L(l,k-i), \qquad \text{(see [14])}.$$

where $L(0,0) = 1$, $L(n,0) = 0$, and $L(n,k) = 0$ for all $k > n$.

In view of (22) and (23), H.K. Kim considered the r-extended central Lah-Bell polynomials $LB_{n,r}^{(C)}(x)$ by the generating function

$$\exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty} LB_{n,r}^{(C)}(x)\frac{t^n}{n!}, \qquad \text{(see [14])}. \tag{53}$$

where $x = 1$, $LB_{n,r}^{(C)}(1) := LB_{n,r}^{(C)}$ are the r-extended central Lah-Bell numbers and $LB_{n,r}^{(C)}(x) = \sum_{k=0}^{n} L_r^{(C)}(n,k)x^k, \quad (n \geq 0)$.

We consider the incomplete r-central Lah-Bell polynomials
$H_{n+r,k+r}^{(r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)$ by the generating function

(54)

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m(\alpha_m-(-1)^m\alpha_m)t^m\right)^k\left(\sum_{h=0}^{\infty}\upsilon_{h+1}t^h\right)^r = \sum_{n=k}^{\infty} H_{n+r,k+r}^{(r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!}.$$

Combining with (41), (42), (47), and (54), we have

$$\begin{aligned}
&H_{n+2r,k+2r}^{(2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\\
&\quad = T_{n+2r,k+2r}^{(2r)}(1!\alpha_1,2!\alpha_2,3!\alpha_3,\cdots:0!\upsilon_1,1!\upsilon_2,2!\upsilon_3\cdots)\\
&\quad = B_{n+2r,k+2r}^{(2r)}(1!\alpha_1,0,\frac{3!}{2^2}\alpha_3,0,\cdots:0!\upsilon_1,1!\upsilon_2,2!\upsilon_3\cdots)\\
&\quad = \sum_{\Lambda(n,k,2r)}\left[\frac{n!}{k_1!k_3!k_5!\cdots}(\alpha_1)^{k_1}\left(\frac{\alpha_3}{2^2}\right)^{k_3}\left(\frac{\alpha_5}{2^4}\right)^{k_5}\cdots\right]\left[\frac{(2r)!}{r_0!r_1!r_2!\cdots}\upsilon_1^{r_0}\upsilon_2^{r_1}\upsilon_3^{r_2}\cdots\right].
\end{aligned} \tag{55}$$

where $\Lambda(n,k,2r)$ denote the set of all nonnegative integers $\{k_i\}_{i\geq 1}$ and $\{r_i\}_{i\geq 0}$ such that

$$\sum_{i\geq 1} k_i = k, \quad \sum_{i\geq 0} r_i = 2r \quad \text{and} \quad \sum_{i\geq 1} i(k_i+r_i) = n.$$

Thus, we have the following theorem.

Theorem 7. *For $n \geq 0$, we have*

$$\begin{aligned} &H^{(2r)}_{n+2r,k+2r}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots) \\ &= \sum_{\Lambda(n,k,2r)}\left[\frac{n!}{k_1!k_3!k_5!\cdots}(\alpha_1)^{k_1}\left(\frac{\alpha_3}{2^2}\right)^{k_3}\left(\frac{\alpha_5}{2^4}\right)^{k_5}\cdots\right]\left[\frac{(2r)!}{r_0!r_1!r_2!\cdots}\upsilon_1^{r_0}\upsilon_2^{r_1}\upsilon_3^{r_2}\right]. \end{aligned}$$

where $\Lambda(n,k,2r)$ denote the set of all nonnegative integers $\{k_i\}_{i\geq 1}$ and $\{r_i\}_{i\geq 0}$ such that

$$\sum_{i\geq 1}k_i = k, \quad \sum_{i\geq 0}r_i = 2r \quad \text{and} \quad \sum_{i\geq 1}i(k_i+r_i) = n.$$

The incomplete r-central Lah-Bell polynomials have the following combinatorial interpretation. Let $\alpha_1,\alpha_2,\cdots$, and $\upsilon_1,\upsilon_2,\cdots$ be any sequences of nonnegative integers.

Then $H^{(2r)}_{n+2r,k+2r}(\alpha_1,2\alpha_2,2^2\alpha_3,\cdots;\upsilon_1,\upsilon_2,\upsilon_3,\cdots)$ enumerates the number of partitions of a set with $(n+r)$ elements into k ordered blocks of odd sizes and r blocks of any sizes satisfying:

· The first r elements are in different blocks,

· Any ordered block of (odd) size i with no elements of the first r elements, can be colored with α_i colors,

· Any ordered block of size i with one element of the first r elements, can be colored with υ_i colors.

The next theorem give the relation between the r-central Lah-Bell numbers and the incomplete r-central Lah-Bell polynomials.

Theorem 8. *For $n \geq k \geq 0$, we have*

$$LB^{(C)}_{n,r} = H^{(2r)}_{n+2r,k+2r}(1,1,\cdots:1,1,\cdots),$$

where $LB^{(C)}_{n,r}$ are the r-central Lah-Bell numbers.

Proof. From (28), (30), (53) and Theorem 8, we obtain

$$\begin{aligned} \sum_{n=k}^{\infty}H^{(2r)}_{n+2r,k+2r}(1,1,\cdots:1,1,\cdots)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h(1-(-1)^h)t^h\right)^k\left(\sum_{m=0}^{\infty}t^m\right)^{2r} \\ &= \sum_{n=k}^{\infty}H_{n,k}(1,1,\cdots,1)\frac{t^n}{n!}\left(\frac{1}{1-t}\right)^{2r} \\ &= \left(\sum_{n=k}^{\infty}L^{(C)}(n,k)\frac{t^n}{n!}\right)\left(\frac{1}{1-t}\right)^{2r} \\ &= \left(\sum_{n=k}^{\infty}\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k\frac{t^n}{n!}\right)\left(\frac{1}{1-t}\right)^{2r} \\ &= exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty}LB^{(C)}_{n,r}\frac{t^n}{n!}. \end{aligned} \tag{56}$$

By comparing with the coefficients of both side of (56), we get the desired result. □

The next theorem give the relation between the central Lah numbers and the incomplete r-central Lah-Bell polynomials.

Theorem 9. *For $n \geq 0$, we observe that*

$$H^{(2r)}_{n+r,k+r}(x,x^2,x^3\cdots;1,x,x^2,\cdots) = \sum_{m=0}^{n}x^n < 2r >_{n-m} L^C(m,k),$$

where $L^C(n,k)$ are the central Lah numbers.

Proof. From (8) and (20), we obtain

$$\begin{aligned}
&\sum_{n=k}^{\infty} H_{n+r,k+r}^{(2r)}(x,x^2,x^3\cdots;1,x,x^2,\cdots)\frac{t^n}{n!}\\
&= \frac{1}{k!}\left(xt+\frac{1}{2^2}x^3t^3+\frac{1}{2^4}x^5t^5+\cdots\right)^k\left(1+xt+x^2t^2+\cdots\right)^{2r}\\
&= \frac{1}{k!}\left(2\left(\frac{1}{2-xt}-\frac{1}{2+xt}\right)\right)^k\left(\frac{1}{1-xt}\right)^{2r} \qquad (57)\\
&= \sum_{m=k}^{\infty} L^C(m,k)\frac{x^m t^m}{m!}\sum_{l=0}^{\infty} <2r>_l \frac{x^l t^l}{l!}\\
&= \sum_{n=k}^{\infty}\sum_{m=0}^{n} x^n <2r>_{n-m} L^C(m,k)\frac{t^n}{n!}.
\end{aligned}$$

By comparing with the coefficients of both side of (57), we get the desired result. □

We observe that

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^k\frac{1}{k!}\left(\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h(\alpha_l-(-1)^h\alpha_h)t^h\right)\left(\sum_{m=0}^{\infty}\upsilon_{m+1}t^m\right)^{2r}\\
&= \sum_{k=0}^{\infty} x^k\sum_{n=k}^{\infty} H_{n+2r,k+2r}^{(2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!} \qquad (58)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k H_{n+2r,k+2r}^{(2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!}\\
&= 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n} x^k H_{n+2r,k+2r}^{(2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!}.
\end{aligned}$$

From (58), we can define the complete r-central Lah-Bell polynomials $H_n^{(C,r)}(x|\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)$, $(n\geq 0)$ which are given by the generating function

(59)

$$exp\left(x\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h(\alpha_h-(-1)^h\alpha_h)t^h\right)\left(\sum_{m=0}^{\infty}\upsilon_{m+1}t^m\right)^r=\sum_{n=0}^{\infty}H_n^{(C,r)}(x|\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!}.$$

In particular, when $x=1$

$$H_n^{(C,r)}(1|\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)=H_n^{(C,r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots). \qquad (60)$$

are called the complete r-central Lah-Bell numbers.

Theorem 10. *For $n\geq 0$, we have*

$$H_n^{(C,2r)}(x|\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)=\sum_{k=0}^{n}x^k H_{n+2r,k+2r}^{(2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots).$$

Proof. By (53) and (59), we observe that

$$(61)\quad \begin{aligned} &exp\left(x\sum_{h=1}^{\infty}\left(\frac{1}{2}\right)^h(\alpha_h-(-1)^h\alpha_h)t^h\right)\left(\sum_{m=0}^{\infty}\upsilon_{m+1}t^m\right)^{2r}\\ &=\sum_{k=0}^{\infty}x^k\frac{1}{k!}\left(\frac{1}{2}\right)^h(\alpha_h-(-1)^h\alpha_h)t^h\Big)^k\left(\sum_{m=0}^{\infty}\upsilon_{m+1}t^m\right)^{2r}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}x^kH^{(2r)}_{n+2r,k+2r}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\right)\frac{t^n}{n!}. \end{aligned}$$

By comparing with coefficients of both side of (61), we get the desired equality. □

The next corollary give the relation between the r-central Lah-Bell polynomials and the complete r-central Lah-Bell polynomials.

Corollary 11. *For $n \geq 0$, we have*

$$H_n^{(C,2r)}(x|1,1,1,\cdots:1,1,1,\cdots)=\sum_{k=0}^{n}LB^C_{n,r}(x).$$

Proof. Combining with Theorem 8 and Theorem 10, we obtain

$$(62)\quad \begin{aligned} &\sum_{n=0}^{\infty}H_n^{(C,2r)}(x|1,1,1,\cdots:1,1,1,\cdots)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(x^k\sum_{k=0}^{n}H^{(2r)}_{n+2r,k+2r}(1,1,\cdots:1,1,\cdots)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}LB^C_{n,r}(x)\frac{t^n}{n!}. \end{aligned}$$

By comparing with the coefficients of both side of (62), we get the desired result. □

Combining (55) and Theorem 10, we get easily next corollaries.

Corollary 12. *For $n \geq 0$, we have*

$$\begin{aligned} \sum_{k=0}^{n}x^{k+2r}H^{(2r)}_{n+2r,k+2r}(1,1,1,\cdots:1,1,1,\cdots)&=\sum_{k=0}^{n}H^{(2r)}_{n+2r,k+2r}(x,x,x,\cdots:x,x,x,\cdots)\\ &=\sum_{k=0}^{n}T^{(2r)}_{n+2r,k+2r}(1!x,2!x,3!x,\cdots:0!x,1!x,2!x,\cdots). \end{aligned}$$

Corollary 13. *For $n \geq 0$, we have*

$$\begin{aligned} &H_n^{(C,2r)}(\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\\ &=\sum\left(\frac{n!}{k_1!k_2!k_3!\cdots}(\alpha_1)^{k_1}\left(\frac{\alpha_3}{2^2}\right)^{k_3}\left(\frac{\alpha_5}{2^4}\right)^{k_5}\cdots\right)\left(\frac{r!}{r_0!r_1!r_2!\cdots}\upsilon_1^{r_0}\upsilon_2^{r_1}\upsilon_3^{r_2}\cdots\right), \end{aligned}$$

where the inner sum over all integers $k_1,k_2,k_3,\cdots \geq 0$ and $r_0,r_1,r_2,\cdots \geq 0$ such that

$$\sum_{i\geq 0}r_i=r,\quad \sum_{i\geq 1}(2i-1)k_{2i-1}+\sum_{i\geq 1}ir_i=n.$$

The next theorem give an explicit formulas of the complete r-central Lah-Bell polynomials.

Theorem 14. *For $n \geq k \geq 0$ with $n-k \equiv 0(mod 2)$, we have*

$$H_n^{(C,2r)}(1\ |\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots) = n!\sum_{l=0}^{n}\Bigg(\sum_{m_1+2m_2+\cdots+lm_l=l}\ \sum_{i_1+i_2+\cdots+i_{2r}=n-l}\frac{1}{m_1!m_2!\cdots m_l!} \times\left(\frac{\alpha_1}{2^0}\right)^{m_1}\left(\frac{0}{2^1}\right)^{m_2}\left(\frac{\alpha_3}{2^2}\right)^{m_3}\cdots\left(\frac{\alpha_l(1-(-1)^l)}{2^{l-1}}\right)^{m_l}\left(\prod_{d=1}^{2s}\upsilon_{i_d+1}\right)\Bigg).$$

Proof. From (40), we have

$$\exp\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right) = \sum_{l=0}^{\infty}\Bigg(\sum_{h_1+2h_2+\cdots+lh_l=l}\frac{1}{h_1!h_2!\cdots h_l!} \times\left(\frac{\alpha_1}{2^0}\right)^{h_1}\left(\frac{0}{2^1}\right)^{h_2}\left(\frac{\alpha_3}{2^2}\right)^{h_3}\cdots\left(\frac{\alpha_l(1-(-1)^l)}{2^{l-1}}\right)^{h_l}\Bigg)t^l. \tag{63}$$

Moreover, we obtain easily

$$\left(\sum_{j=0}^{\infty}\upsilon_{j+1}t^j\right)^{2s} = \sum_{i=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{2r}=i}\left(\prod_{d=1}^{2s}\upsilon_{i_d+1}\right)t^i. \tag{64}$$

By (63) and (64), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}H_n^{(C,2r)}(1\ |\alpha_1,\alpha_2,\cdots:\upsilon_1,\upsilon_2,\cdots)\frac{t^n}{n!}\\
&= \exp\left(\sum_{h=1}^{\infty}\frac{1}{2^h}(\alpha_h-(-1)^h\alpha_h)t^h\right)\left(\sum_{j=0}^{\infty}\upsilon_{j+1}t^j\right)^{2s}\\
&= \sum_{n=0}^{\infty}n!\sum_{l=0}^{n}\Bigg(\sum_{h_1+2h_2+\cdots+lh_l=l}\ \sum_{i_1+i_2+\cdots+i_{2r}=n-l}\frac{1}{h_1!h_2!\cdots h_l!}\\
&\quad\times\left(\frac{\alpha_1}{2^0}\right)^{h_1}\left(\frac{0}{2^1}\right)^{h_2}\left(\frac{\alpha_3}{2^2}\right)^{h_3}\cdots\left(\frac{\alpha_l(1-(-1)^l)}{2^{l-1}}\right)^{h_l}\left(\prod_{d=1}^{2s}\upsilon_{i_d+1}\right)\Bigg)\frac{t^n}{n!}.
\end{aligned} \tag{65}$$

Therefore, by comparing with coefficients of both side of (65), we get the desired result. □

4. Conclusion

Recently, the author introduced the central Lah-numbers and r-central Lah-numbers and studied their interesting properties [13, 14]. Following this study, in Section 2, both the complete and incomplete central Lah-Bell polynomials as multivariate forms of both central Lah-numbers and central Lah-Bell polynomials, respectively were introduced. the author demonstrated that the special cases of these two polynomials are the central Lah-numbers and the central Lah-Bell numbers respectively, in Theorems 3 and 4. Moreover, we provided explicit formulas for computing the central Lah-Bell numbers by employing the Kolbig-coeffey equation in Theorem 1 and the complete central Lah-Bell polynomials in Theorem 6.

As for the Section 3, both the complete and incomplete r-central Lah-Bell polynomials as multivariate forms of both r-central Lah-Bell polynomials and r-central Lah-numbers, respectively were also introduced. Notably, the author have shown that the special cases of those two polynomials are the r-extended central Lah-Bell numbers and the central Lah-numbers in Theorem 8, 9 and corollary 11. Additionally, the author also derived basic formulas for computing both the r-extended central complete and incomplete Lah-Bell polynomials in Theorems 7 and 14, respectively.

In conclusion, in relation to early research conducted, we brought the reader's attention to both complete and incomplete Bell polynomials potential for applications of probability theory, combinatorial theory and number theory [3-5, 7, 8, 20, 25, 26, 29, 32]. Hence, for future projects, we would like to conduct research into some potential applications of the numbers and polynomials derived in this paper.

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